

POLYNOMIAL EXPRESSIONS OF p -ARY AUCTION FUNCTIONS

SHIZUO KAJI, TOSHIAKI MAENO, KOJI NUIDA, AND YASUhide NUMATA

ABSTRACT. Let \mathbb{F}_p be the finite field of prime order p . For any function $f: \mathbb{F}_p^n \rightarrow \mathbb{F}_p$, there exists a unique polynomial over \mathbb{F}_p having degree at most $p - 1$ with respect to each variable which coincides with f . We call it the minimal polynomial of f . It is in general a non-trivial task to find a concrete expression of the minimal polynomial of a given function, which has only been worked out for limited classes of functions in the literature. In this paper, we study minimal polynomial expressions of several functions that are closely related to some practically important procedures such as auction and voting.

1. INTRODUCTION

Let p be a prime and \mathbb{F}_p the finite field of order p . It is well-known that any function $f: \mathbb{F}_p^n \rightarrow \mathbb{F}_p$ can be expressed as a polynomial $P(x_1, \dots, x_n)$ with coefficients in \mathbb{F}_p , and such a polynomial is unique if its degree with respect to each variable is restricted to be at most $p - 1$; we call the unique polynomial P the *minimal polynomial* of the function f . In theory, it is easy by Fermat's Little Theorem to see that the polynomial P is given by $\sum_{(a_1, \dots, a_n) \in \mathbb{F}_p^n} f(a_1, \dots, a_n) \delta_{a_1}(x_1) \cdots \delta_{a_n}(x_n)$, where $\delta_{a_i}(x_i) = 1 - (x_i - a_i)^{p-1}$ is the minimal polynomial for the Kronecker delta. This expression, however, has two shortcomings; it relies on the (often implicit) values $f(a_1, \dots, a_n)$ of the function, and it in general contains many redundant terms to be cancelled out. As a result, it remains a non-trivial task to obtain an *explicit* and concise minimal polynomial expression for a given concrete function f . For example, Sturdivant and Frandsen [8, Theorems 9.1(a) and 11.2] showed that the carry function in multiplication of p -ary integers is expressed by using number-theoretic objects such as the Bernoulli numbers and Wilson's quotient (see also [5] for a different approach to the result and an expression of the carry function in the case of addition of p -ary integers). As this previous result suggests, the problem of computing minimal polynomial expressions of certain functions can lead to interesting theoretical results connecting different fields of mathematics.

On the other hand, this problem has potential applications in cryptography as well. There was recently a breakthrough in the area of cryptography, namely the discovery of *fully homomorphic encryption* (see [3, 7] for survey). One can compute *in an encrypted form* both addition and multiplication over the two-element field \mathbb{F}_2 (see [4], etc.) and over even larger finite prime fields \mathbb{F}_p for $p > 2$ (see [6]). It follows that one can compute any function provided the function is explicitly written as a polynomial over \mathbb{F}_p . For example, a recent work [2] on practical cryptographic systems based on fully homomorphic encryption relies on a recursive polynomial expression of the comparison function for two binary integers. To develop such practical systems, "efficient" polynomial expressions of various functions are useful, and in particular, the minimal degree condition is important since encrypted multiplication is in general computationally much more expensive than encrypted addition.

In this paper, we study minimal polynomial expressions of a certain kind of functions specified below. They are relevant to some practical procedures such as auction and voting. We chiefly discuss the max function that takes an element of \mathbb{F}_p^n as input and returns the largest value among them, and the argmax function that returns the least index of the largest component(s) in the input vector in \mathbb{F}_p^n . Here we clarify that, the finite field \mathbb{F}_p is naturally identified with the subset $\{0, 1, \dots, p - 1\}$ of integers, and comparison of elements (e.g., in the function max) is performed in the latter, while addition and multiplication are done in the former. The output of argmax is an integer that may exceed

Date: March 24, 2017.

2010 Mathematics Subject Classification. 68R05, 12Y05.

Key words and phrases. Polynomial expression of functions, finite fields, cryptography.

The fourth named author was partially supported by KAKENHI, Grant-in-Aid for Young Scientists (B) JP25800009.

the range of the field \mathbb{F}_p when $n \geq p$. To handle this, we introduce an \mathbb{F}_p -valued function $\text{argmax}^{(r)}$ that returns the r -th digit of the p -ary expansion of argmax .

In §2, we define and give the minimal polynomial for the “low-pass filtering function” $L_t(x)$ and the Kronecker delta function $\delta_t(x)$, which are used as building blocks in the later sections. In §3, we give a minimal polynomial expression of the function max in terms of $L_t(x)$ and $\delta_t(x)$. However, these general expressions contain many terms. We derive more concise forms for $p = 2$ and 3 (Corollary 3.2 and Proposition 3.3). A duality between max and min allows us to deduce corresponding formulae for min . §4 is devoted to the study of the argmax function. First, we provide a way of reducing the computation for $\text{argmax}^{(r)}$ with any $r \geq 0$ to the computation for $\text{argmax}^{(0)}$ by utilising the result on the function max . We also provide a recursive formula for $\text{argmax}^{(r)}$ with respect to the input length. They are used to derive minimal polynomial expressions of $\text{argmax}^{(r)}$ when $p = 2$ and of $\text{argmax}^{(0)}$ for $p = 3$ and $n = 3$ (Propositions 4.3, 4.4, and 4.6). The recursive formula for $\text{argmax}^{(r)}$ relies on the (minimal) polynomial expression of $\text{argmax}^{(0)}$ with input length of two. We give a minimal polynomial expression of $\text{argmax}^{(0)}$ for $n = 2$ and any p in §5, which also yields a minimal polynomial expression of max with $n = 2$ (Proposition 5.2 and Theorem 5.5).

In §6, we introduce and study two more functions that are also relevant to our problem. We recall that the definition of argmax enforces the function to always output the first index when there are ties in the input vectors; this then loses the information on the other largest components of the input. To remedy this situation, we introduce the function “ $\text{ismax}(y; x)$ ” that returns if the maximum value among the components of the input vector x is equal to the other input value $y \in \mathbb{F}_p$, and “ $\text{nummax}(x)$ ” that returns the number of inputs which attain the tied maximum. Then, similarly to the cases of max and argmax , we provide a general formula for the minimal polynomial expression of ismax and nummax in terms of the low-pass filtering functions and the Kronecker delta functions, and also compute concise forms of minimal polynomial expressions of ismax for $p = 2$ and 3 , and of nummax for $p = 2$.

We conclude with a possible extension of our result to a multi-digit setting in §7.

Acknowledgement. The authors would like to thank Takuro Abe for fruitful discussions.

2. NOTATION AND BASIC FUNCTIONS

In this section, we fix some notations used throughout the paper. A vector x of length n over the field \mathbb{F}_p is denoted by $(x_0, x_1, \dots, x_{n-1})$. We introduce a linear ordering $<$ on \mathbb{F}_p via the natural identification of it with the subset $\{0, 1, \dots, p-1\}$ of \mathbb{Z} (with the usual ordering $<$). We denote by $e_i(x)$ the i -th elementary symmetric polynomial of x_0, x_1, \dots, x_{n-1} so that $\prod_{i=0}^{n-1} (1 + x_i) = \sum_{i=0}^n e_i(x)$.

For a logical formula P with free variable x , we define its *truth function* by

$$\chi_P(x) = \begin{cases} 1 & (P(x) \text{ is true}) \\ 0 & (\text{otherwise}) \end{cases}$$

which is often abbreviated as $\chi_P(x) = \chi(P)$. We frequently use the same symbol for a function and its polynomial expression.

Example 2.1. For $t \in \mathbb{F}_p$, the minimal polynomial for the *delta function* $\delta_t(x) = \chi(x = t)$ is given by

$$\delta_t(x) = 1 - (x - t)^{p-1} = - \prod_{i=1}^{p-1} (x - t + i) ,$$

which follows from Fermat’s Little Theorem. Similarly, the minimal polynomial for the *low-pass function* $L_t(x) = \chi(x < t)$ is given by

$$L_t(x) = \sum_{0 \leq k < t} \delta_k(x) = \sum_{0 \leq k < t} (1 - (x - k)^{p-1}) .$$

For an integer $k \geq 0$, its r -th digit in the p -ary expansion is denoted by $k^{(r)}$; that is, $k = \sum_{r=0}^{\infty} k^{(r)} p^r$ with $k^{(r)} \in \{0, 1, \dots, p-1\}$ for each r .

3. POLYNOMIAL EXPRESSIONS OF THE \max AND THE \min FUNCTIONS

For a vector $x = (x_0, x_1, \dots, x_{n-1}) \in \mathbb{F}_p^n$, let $\max(x)$ (respectively, $\min(x)$) denote the maximum (respectively, minimum) among the n values x_0, x_1, \dots, x_{n-1} .

Using the functions in Example 2.1, we immediately obtain the minimal polynomial of \max .

Proposition 3.1. The minimal polynomial of \max is given by

$$\max(x) = \sum_{1 \leq t \leq p-1} \chi(x_i \geq t \text{ for some } i) = \sum_{1 \leq t \leq p-1} \left(1 - \prod_{i=0}^{n-1} L_t(x_i) \right).$$

In particular, when $p = 2$ this simplifies:

Corollary 3.2. The minimal polynomial of $\max(x)$ for $p = 2$ is given by

$$\max(x) = \prod_{i=0}^{n-1} (1 + x_i) - 1 = \sum_{i=1}^n e_i(x).$$

However when $p > 2$, the expression in Proposition 3.1 consists of a lot of terms. We now compute a more concise expression for $p = 3$.

First we note that $\max(x) + 1 = 0$ if $x_i = p - 1$ for some x_i . This implies that the minimal polynomial of $\max(x) + 1$ has $1 + x_i$ as a factor for every i . Therefore, we have

$$\max(x) = f_n(x) \prod_{i=0}^{n-1} (1 + x_i) - 1 = f_n(x) \sum_{i=0}^n e_i(x) - 1$$

for some polynomial $f_n(x)$ in which each variable x_i has degree at most $p - 2$. In particular, this observation yields another proof of Corollary 3.2 (where $p = 2$). For the case $p = 3$, we have the following result:

Proposition 3.3. When $p = 3$, a minimal polynomial expression for $\max(x)$ is given by:

$$\max(x) = \sum_{i=0}^{\lfloor n/2 \rfloor} e_{2i}(x) \sum_{i=0}^n e_i(x) - 1.$$

Proof. Denote the right hand side by $P(x)$. As the minimality condition on the degree is satisfied for $P(x)$, it suffices to verify $\max(x) = P(x)$ for any $x \in \mathbb{F}_p^n$. When $\max(x) = 2$, there exists i such that $x_i = 2$. This implies $\prod_{i=0}^{n-1} (1 + x_i) = \sum_{i=0}^n e_i(x) = 0$ and $P(x) = -1 = 2$. Notice that

$$\sum_{i=0}^{\lfloor n/2 \rfloor} e_{2i}(x) \sum_{i=0}^n e_i(x) = 2 \left(\prod_{i=0}^{n-1} (1 + x_i)^2 + \prod_{i=0}^{n-1} (1 - x_i^2) \right)$$

by the definition of $e_i(x)$ and the fact $2^{-1} = 2$ in \mathbb{F}_3 . When $\max(x) = 1$, as $\prod_{i=0}^{n-1} (1 + x_i)^2 = 1$ and $\prod_{i=0}^{n-1} (1 - x_i^2) = 0$, we have $P(x) = 2(1 + 0) - 1 = 1$. When $\max(x) = 0$, as $\prod_{i=0}^{n-1} (1 + x_i)^2 = 1$ and $\prod_{i=0}^{n-1} (1 - x_i^2) = 1$, we have $P(x) = 2(1 + 1) - 1 = 0$. \square

To obtain a minimal polynomial expression for \min , we exploit a duality between \max and \min . Define an *involution* on \mathbb{F}_p by $\bar{x} = p - 1 - x$ and extend it coordinate-wisely on \mathbb{F}_p^n . Then, we have $\overline{\min(x)} = \max(\bar{x})$ for any $x \in \mathbb{F}_p^n$. Thus, a minimal polynomial expression for \max converts to one of \min and vice versa. For example, Corollary 3.2 and Proposition 3.3 imply the following:

Corollary 3.4. When $p = 2$, a minimal polynomial expression for \min is given by

$$\min(x) = \prod_{i=0}^{n-1} x_i = e_n(x).$$

When $p = 3$, a minimal polynomial expression for \min is given by

$$\min(x) = \prod_{i=0}^{n-1} x_i^2 + \prod_{i=0}^{n-1} x_i(1 - x_i) = e_n \left(1 + \sum_{i=1}^n (-1)^i e_i + e_n \right).$$

For the next case of $p = 5$, minimal polynomial expressions of $\max(x)$ for small values of n in terms of elementary symmetric polynomials can be determined by direct calculation:

Example 3.5. When $p = 5$, the following are minimal polynomial expressions.

- $\max(x_0, x_1) = (1 + e_1 + e_2)(1 + 2e_1^2e_2 + 4e_1e_2 + e_2) - 1$
- $\max(x_0, x_1, x_2) = (1 + e_1 + e_2 + e_3)(1 + 2e_1^2e_2 + e_1e_2e_3 + 2e_1e_3^2 + e_2^2e_3 + 2e_2e_3^2 + 4e_1e_2 + 3e_1e_3 + e_2e_3 + 3e_2^2 + e_2) - 1$

However, it seems to be difficult to obtain a general formula (such as Proposition 3.3) for $p \geq 5$. The function $\max(x)$ with $n = 2$ for any p will be revisited in §4.

Remark 3.6. The function $\max(x)$ is a symmetric function (in variables x_0, x_1, \dots, x_{n-1}), and satisfies $\max(x, 0) = \max(x)$ and an “associativity” in the following sense:

$$\max(x_0, x_1, \dots, x_{n-1}, x_n) = \max(\max(x_0, \dots, x_{n-1}), x_n) = \max(x_0, \max(x_1, \dots, x_n)) .$$

By using this property recursively, a minimal polynomial expression of the \max function with two variables (i.e., for the case $n = 2$) yields a polynomial expression of \max with any number of variables (i.e., for any n). However, the polynomial thus obtained is *not* the minimal polynomial in general.

4. POLYNOMIAL EXPRESSIONS OF THE argmax FUNCTION

Let $\operatorname{argmax}(x)$ be the least integer i such that $x_i = \max(x)$. Note that $\operatorname{argmax}(x)$ takes values in $\{0, 1, \dots, n-1\}$ so we define for $r \geq 0$

$$\operatorname{argmax}^{(r)}: \mathbb{F}_p^n \rightarrow \mathbb{F}_p, \quad \operatorname{argmax}^{(r)}(x) = \operatorname{argmax}(x)^{(r)},$$

where $\operatorname{argmax}^{(r)}(x)$ is the r -th digit in the p -ary expansion of $\operatorname{argmax}(x)$.

Again using the functions in Example 2.1, we immediately obtain the minimal polynomial of $\operatorname{argmax}^{(r)}$.

Proposition 4.1. The minimal polynomial for $\operatorname{argmax}^{(r)}$ is given by

$$\begin{aligned} \operatorname{argmax}^{(r)}(x) &= \sum_{i=0}^{n-1} i^{(r)} \cdot \chi(\operatorname{argmax}(x) = i) \\ &= \sum_{i=0}^{n-1} i^{(r)} \left(\sum_{0 \leq t \leq p-1} \left(\delta_t(x_i) \prod_{0 \leq j < i} L_t(x_j) \prod_{i < k \leq n-1} L_{t+1}(x_k) \right) \right) . \end{aligned}$$

Remark 4.2. Let $\operatorname{argmin}(x)$ be the function which returns the least index i with $\min(x) = x_i$. A minimal polynomial expression of argmin is obtained from one of argmax via the duality $\operatorname{argmin}(x) = \operatorname{argmax}(\bar{x})$ similarly to the case of \min discussed in §3.

Observe by definition of the function $\operatorname{argmax}^{(r)}$ that

$$(4.1) \quad \operatorname{argmax}^{(r)}(x) = \operatorname{argmax}^{(0)}(\max(x_0, x_1, \dots, x_{p^r-1}), \dots, \max(x_{i \cdot p^r}, x_{i \cdot p^r+1}, \dots, x_{(i+1) \cdot p^r-1}), \dots) ,$$

$$(4.2) \quad \begin{aligned} \operatorname{argmax}^{(r)}(x_0, \dots, x_{n-1}, x_n) &= \operatorname{argmax}^{(r)}(x_0, \dots, x_{n-1}) \cdot (1 - \operatorname{argmax}^{(0)}(\max(x_0, \dots, x_{n-1}), x_n)) \\ &\quad + n^{(r)} \cdot \operatorname{argmax}^{(0)}(\max(x_0, \dots, x_{n-1}), x_n) . \end{aligned}$$

The second equation follows from

$$\operatorname{argmax}(x_0, \dots, x_{n-1}, x_n) = \begin{cases} \operatorname{argmax}(x_0, \dots, x_{n-1}) & \text{if } \max(x_0, \dots, x_{n-1}) \geq x_n, \\ n & \text{if } \max(x_0, \dots, x_{n-1}) < x_n. \end{cases}$$

These formulae yield a (in general, not minimal) polynomial expression of $\operatorname{argmax}^{(r)}(x)$ from those of $\operatorname{argmax}^{(0)}(x)$ with $n = 2$ and $\max(x)$.

4.1. **The case $p = 2$.** When $p = 2$, we can derive a minimal polynomial expression of $\text{argmax}^{(r)}$.

Proposition 4.3. When $p = 2$ and $k, r \geq 0$, the following are minimal polynomial expressions:

$$\begin{aligned} \text{argmax}^{(r)}(x_0, x_1, \dots, x_{(2k+2)2^r-1}) &= \text{argmax}^{(r)}(x_0, x_1, \dots, x_{(2k+1)2^r-1}) \\ &= \sum_{i=0}^k (1+x_0)(1+x_1) \cdots (1+x_{(2i+1)2^r-1}) \left(\prod_{j=(2i+1)2^r}^{(2i+2)2^r-1} (1+x_j) - 1 \right) \\ &= \sum_{i=1}^{2k+2} (1+x_0)(1+x_1) \cdots (1+x_{i \cdot 2^r-1}) , \\ \text{argmax}^{(r)}(x_0, x_1, \dots, x_{n-1}) &= \text{argmax}^{(r)}(x_0, x_1, \dots, x_{n-1}, 0) . \end{aligned}$$

Proof. Notice that $\text{argmax}^{(0)}(x) = 1$ if and only if there is an odd index i satisfying that $x_j = 0$ for every $j < i$ and $x_i = 1$. So we have

$$\text{argmax}^{(0)}(x_0, x_1, \dots, x_{2k+2}) = \text{argmax}^{(0)}(x_0, x_1, \dots, x_{2k+1}) = \sum_{i=0}^k (1+x_0)(1+x_1) \cdots (1+x_{2i})x_{2i+1} .$$

Combining this with (4.1) and Proposition 3.2, we obtain the first formula (note that the characteristic is now $p = 2$). The second formula follows from the fact $\text{argmax}(x, 0) = \text{argmax}(x)$. \square

We can also use (4.2) to give another formula:

Proposition 4.4. When $p = 2$ and $r \geq 0$, a minimal polynomial expression of $\text{argmax}^{(r)}(x)$ is given as follows:

- (1) $\text{argmax}^{(r)}(x_0) = 0$,
- (2) If $2^{r+1}k + 2^r \leq n < 2^{r+1}(k+1)$ for an integer $k \geq 0$ (i.e., $n^{(r)} = 1$), then we have

$$\text{argmax}^{(r)}(x_0, x_1, \dots, x_n) = \sum_{j=0}^k \sum_{i=0}^{\min(2^r-1, n-2^{r+1}j-2^r)} (1+x_0)(1+x_1) \cdots (1+x_{2^{r+1}j+2^r+i-1})x_{2^{r+1}j+2^r+i} ,$$

- (3) If $2^{r+1}k \leq n < 2^{r+1}k + 2^r$ for an integer $k \geq 0$ (i.e., $n^{(r)} = 0$), then we have

$$\text{argmax}^{(r)}(x_0, x_1, \dots, x_n) = \text{argmax}^{(r)}(x_0, x_1, \dots, x_{n-1}) .$$

Proof. Let $f(x) = f(x_0, \dots, x_n)$ denote the right-hand side of the claimed equality in the statement (we define $f(x) = 0$ when $n = 0$). First we note that, in the present case $p = 2$ we have

$$\text{argmax}^{(r)}(x_0, \dots, x_{n-1}) \text{argmax}^{(0)}(\max(x_0, \dots, x_{n-1}), x_n) = 0 ,$$

since the only possibility to satisfy $\text{argmax}^{(0)}(\max(x_0, \dots, x_{n-1}), x_n) = 1$ is that $x_i = 0$ for every $i < n$ and $x_n = 1$, which then implies $\text{argmax}(x_0, \dots, x_{n-1}) = 0$. Therefore, the recursive formula (4.2) now becomes

$$\text{argmax}^{(r)}(x_0, \dots, x_{n-1}, x_n) = \text{argmax}^{(r)}(x_0, \dots, x_{n-1}) + n^{(r)} \cdot \text{argmax}^{(0)}(\max(x_0, \dots, x_{n-1}), x_n) .$$

It then suffices to show that $f(x)$ instead of $\text{argmax}^{(r)}(x)$ also satisfies the same recursive formula. This is obvious when n satisfies the condition for the second case in the statement.

From now on, we focus on the other case where n satisfies the condition for the first case in the statement. Since direct computation shows $\text{argmax}^{(0)}(x_0, x_1) = (x_0 + 1)x_1$, by Proposition 3.2 we have

$$\text{argmax}^{(0)}(\max(x_0, \dots, x_{n-1}), x_n) = (\max(x_0, \dots, x_{n-1}) + 1)x_n = (1+x_0)(1+x_1) \cdots (1+x_{n-1})x_n ,$$

therefore the recursive formula now becomes

$$\text{argmax}^{(r)}(x_0, \dots, x_{n-1}, x_n) = \text{argmax}^{(r)}(x_0, \dots, x_{n-1}) + (1+x_0)(1+x_1) \cdots (1+x_{n-1})x_n .$$

If $n \neq 2^{r+1}k + 2^r$, then $n-1$ also satisfies the same condition as n with the same integer k , and now we indeed have $f(x_0, \dots, x_{n-1}, x_n) = f(x_0, \dots, x_{n-1}) + (1+x_0)(1+x_1) \cdots (1+x_{n-1})x_n$ by the definition of f (note that $\min(2^r-1, n-2^{r+1}k-2^r) = n-2^{r+1}k-2^r$ in this case). On the other hand, if $n = 2^{r+1}k + 2^r$, then we have $f(x_0, \dots, x_{n-1}) = f(x_0, \dots, x_{2^{r+1}k-1})$ by the definition of f for the second case in the

statement, while $2^{r+1}k - 1$ satisfies the condition for the first case in the statement with $k - 1$ playing the role of k . This implies that $f(x_0, \dots, x_{n-1}, x_n) = f(x_0, \dots, x_{n-1}) + (1 + x_0)(1 + x_1) \cdots (1 + x_{n-1})x_n$ also holds in this case by the definition of f (note that now $\min(2^r - 1, n - 2^{r+1}k - 2^r) = n - 2^{r+1}k - 2^r = 0$). Hence f satisfies the desired recursive formula in any case, completing the proof. \square

For Proposition 4.4, by noting that $x_{2^{r+1}j+2^r+i} = (1 + x_{2^{r+1}j+2^r+i}) - 1$ and now the characteristic is $p = 2$, the formula given there can be rewritten as follows.

Corollary 4.5. Let $S(r, n)$ be the set of integers defined by

$$\begin{aligned} S(r, n) := & \{2^{r+1}k + 2^r - 1 \mid 0 \leq k < 2^{-r-1}(n + 1 - 2^r)\} \\ & \cup \{2^{r+1}k - 1 \mid 1 \leq k < 2^{-r-1}(n + 1 - 2^r)\} \\ & \cup \{\min(n, 2^{r+1}(\lfloor (n - 2^r)/2^{r+1} \rfloor + 1) - 1)\} . \end{aligned}$$

A minimal polynomial expression of $\operatorname{argmax}^{(r)}$ is given by

$$\operatorname{argmax}^{(r)}(x_0, x_1, \dots, x_n) = \sum_{i \in S(r, n)} (1 + x_0)(1 + x_1) \cdots (1 + x_i) .$$

4.2. The case $p = 3$. When $p = 3$, a direct computation shows

$$\operatorname{argmax}^{(0)}(x_0, x_1) = x_1(1 + x_0)(x_1 - x_0).$$

Combining this with the following formula from Proposition 3.3

$$\max(x_0, \dots, x_n) = \prod_{i=0}^n (1 + x_i) \cdot \left(1 + \sum_{i \geq 1} e_{2i}(x_0, \dots, x_n)\right) - 1 ,$$

we obtain by (4.2):

$$\begin{aligned} & \operatorname{argmax}^{(r)}(x_0, \dots, x_n, x_{n+1}) \\ &= \operatorname{argmax}^{(r)}(x_0, \dots, x_n) \left(1 - x_{n+1} \prod_{i=0}^n (1 + x_i) \cdot \left(1 + \sum_{i \geq 1} e_{2i}(x_0, \dots, x_n)\right) \right. \\ & \quad \cdot \left. \left(1 + x_{n+1} - \prod_{i=0}^n (1 + x_i) \cdot \left(1 + \sum_{i \geq 1} e_{2i}(x_0, \dots, x_n)\right)\right)\right) \\ & \quad + (n + 1)^{(r)} \cdot \left(x_{n+1} \prod_{i=0}^n (1 + x_i) \cdot \left(1 + \sum_{i \geq 1} e_{2i}(x_0, \dots, x_n)\right) \right. \\ & \quad \cdot \left. \left(1 + x_{n+1} - \prod_{i=0}^n (1 + x_i) \cdot \left(1 + \sum_{i \geq 1} e_{2i}(x_0, \dots, x_n)\right)\right)\right) . \end{aligned}$$

Although this formula does not yield a minimal polynomial expression for $\operatorname{argmax}^{(r)}$ directly, we can still compute one at least when n is not too large. For example, when the input vector has length 3 (hence it suffices to consider $r = 0$ only), the formula above (with $r = 0$ and $n = 1$) becomes

$$\begin{aligned} & \operatorname{argmax}^{(0)}(x_0, x_1, x_2) \\ &= \operatorname{argmax}^{(0)}(x_0, x_1) \left(1 - x_2(1 + x_0)(1 + x_1)(1 + x_0x_1) \left(1 + x_2 - (1 + x_0)(1 + x_1)(1 + x_0x_1)\right)\right) \\ & \quad + 2 \cdot \left(x_2(1 + x_0)(1 + x_1)(1 + x_0x_1) \left(1 + x_2 - (1 + x_0)(1 + x_1)(1 + x_0x_1)\right)\right) \\ &= x_1(1 + x_0)(x_1 - x_0) \\ & \quad - \left(x_1(1 + x_0)(x_1 - x_0) + 1\right) \left(x_2(1 + x_0)(1 + x_1)(1 + x_0x_1) \left(1 + x_2 - (1 + x_0)(1 + x_1)(1 + x_0x_1)\right)\right) . \end{aligned}$$

A straightforward expansion of the polynomial in the right-hand side yields

$$\begin{aligned} \operatorname{argmax}^{(0)}(x_0, x_1, x_2) = & -x_0x_1 - x_0^2x_1 + x_1^2 + x_0x_1^2 \\ & + x_2(x_0 + x_0^2 + x_1 + x_0x_1 + x_0^2x_1 - x_0^4x_1 + x_1^2 - x_0x_1^2 + x_0^3x_1^2 + x_0^5x_1^2 + x_1^3 - x_0^3x_1^3 + x_0^4x_1^3 - x_0^5x_1^3 - x_0^6x_1^3 \\ & + x_1^4 + x_0^2x_1^4 - x_0^4x_1^4 - x_0^5x_1^4 + x_0^6x_1^4 - x_0x_1^5 - x_0^2x_1^5 - x_0^3x_1^5 - x_0^4x_1^5 - x_0^5x_1^5 - x_0^6x_1^5 + x_0^2x_1^6 + x_0^5x_1^6) \\ & + x_2^2(-1 - x_0 - x_1 - x_0x_1 + x_0^2x_1 + x_0^3x_1 - x_1^2 + x_0x_1^2 + x_0^2x_1^2 + x_0^4x_1^2 \\ & - x_1^3 + x_0^2x_1^3 + x_0^3x_1^3 + x_0^4x_1^3 - x_0x_1^4 + x_0^2x_1^4 - x_0^3x_1^4) \end{aligned}$$

and, by applying the relations $x_0^3 \equiv x_0$ and $x_1^3 \equiv x_1$ several times (where \equiv means equivalence as functions over \mathbb{F}_p), we finally obtain:

Proposition 4.6. A minimal polynomial expression of $\operatorname{argmax}^{(0)}(x_0, x_1, x_2)$ for $p = 3$ is given by

$$\operatorname{argmax}^{(0)}(x_0, x_1, x_2) = 2(x_0x_1^2x_2 + x_1^2x_2^2 + x_1^2x_2 + 2x_1x_2^2 + x_0x_1 + 2x_0x_2 + 2x_1^2 + x_1x_2 + x_2^2)(x_0 + 1) .$$

5. POLYNOMIAL EXPRESSIONS OF THE MAX AND ARGMAX FUNCTIONS FOR TWO VARIABLES

First we note that $x_0 < x_1$ if and only if $\overline{x_0}$ and x_1 add up to an equal or greater integer than p when considered as integers; that is, $\operatorname{argmax}^{(0)}(x_0, x_1)$ is equal to the carry by the p -ary addition of two single-digit values $\overline{x_0}$ and x_1 to the next digit. A minimal polynomial expression of this carry function, denoted by φ_1 , has been determined in [5, 6]:

Lemma 5.1 ([5, 6]). For $y_0, y_1 \in \mathbb{F}_p$, we have

$$\varphi_1(y_0, y_1) = \sum_{d=1}^{p-1} (-1)^d d^{-1} y_0(y_0 - 1) \cdots (y_0 - d + 1) y_1(y_1 - 1) \cdots (y_1 - (p - d) + 1) ,$$

where the d^{-1} in the right-hand side means the inverse of d as an element of \mathbb{F}_p .

Combining this with $\operatorname{argmax}^{(0)}(x_0, x_1) = \varphi_1(\overline{x_0}, x_1)$, we obtain:

Proposition 5.2. When $n = 2$, a minimal polynomial expression of $\operatorname{argmax}^{(0)}(x_0, x_1)$ is given by

$$\operatorname{argmax}^{(0)}(x_0, x_1) = \sum_{d=1}^{p-1} d^{-1} (x_0 + 1)(x_0 + 2) \cdots (x_0 + d) x_1(x_1 - 1) \cdots (x_1 - (p - d) + 1) .$$

Example 5.3. By using Proposition 5.2 (or direct calculation), we have the following minimal polynomial expressions of $\operatorname{argmax}^{(0)}(x_0, x_1)$ for small primes.

- When $p = 2$, $\operatorname{argmax}^{(0)}(x_0, x_1) = (x_0 + 1)x_1$.
- When $p = 3$, $\operatorname{argmax}^{(0)}(x_0, x_1) = -(x_0 + 1)(x_0 - x_1)x_1$.
- When $p = 5$, $\operatorname{argmax}^{(0)}(x_0, x_1) = -(x_0 + 1)(x_0^2 - x_0x_1 + x_0 + x_1^2)(x_0 - x_1)x_1$.
- When $p = 7$, $\operatorname{argmax}^{(0)}(x_0, x_1) = -(x_0^4 + 5x_0^3x_1 + 2x_0^3 + 3x_0^2x_1^2 + x_0^2x_1 + 4x_0^2 + 5x_0x_1^3 + 6x_0x_1^2 + 3x_0 + x_1^4)(x_0 + 1)(x_0 - x_1)x_1$.

We also have the following relation between max and argmax deduced from their definitions:

Lemma 5.4. We have $\max(x) = \sum_{i=0}^{n-1} x_i \cdot \chi(\operatorname{argmax}(x) = i)$. In particular, we have

$$\max(x_0, x_1) = x_0 \cdot (1 - \operatorname{argmax}(x_0, x_1)) + x_1 \cdot \operatorname{argmax}(x_0, x_1) .$$

A straightforward substitution of the result of Proposition 5.2 into the right-hand side of Lemma 5.4 yields an almost, but not yet minimal, polynomial expression of $\max(x_0, x_1)$. This expression can be converted to a minimal polynomial expression.

Theorem 5.5. When $p \geq 3$, we have the following minimal polynomial expression of $\max(x_0, x_1)$:

$$\begin{aligned} \max(x_0, x_1) = & (x_1 - x_0) \sum_{d=2}^{p-2} d^{-1} (x_0 + 1)(x_0 + 2) \cdots (x_0 + d) x_1(x_1 - 1) \cdots (x_1 - (p - d) + 1) \\ & + x_0 + (x_0 + 1)^2(1 - (x_1 + 1)^{p-1}) + (1 - x_0^{p-1})x_1^2 . \end{aligned}$$

Proof. Throughout the proof, a notation $f \equiv g$ means that f and g define an identical function on \mathbb{F}_p . First, since $p \geq 3$, Proposition 5.2 implies

$$\begin{aligned} & x_0 \operatorname{argmax}(x_0, x_1) \\ &= x_0 \sum_{d=1}^{p-1} d^{-1}(x_0+1)(x_0+2) \cdots (x_0+d)x_1(x_1-1) \cdots (x_1-(p-d)+1) \\ &= x_0 \sum_{d=2}^{p-2} d^{-1}(x_0+1)(x_0+2) \cdots (x_0+d)x_1(x_1-1) \cdots (x_1-(p-d)+1) \\ &\quad + x_0(x_0+1)x_1(x_1-1) \cdots (x_1-(p-2)) - x_0(x_0+1)(x_0+2) \cdots (x_0+p-1)x_1 \end{aligned}$$

and we have $x_0(x_0+1)(x_0+2) \cdots (x_0+p-1)x_1 \equiv 0$ for the last term above. Similarly, we have

$$\begin{aligned} & x_1 \operatorname{argmax}(x_0, x_1) \\ &= x_1 \sum_{d=1}^{p-1} d^{-1}(x_0+1)(x_0+2) \cdots (x_0+d)x_1(x_1-1) \cdots (x_1-(p-d)+1) \\ &= x_1 \sum_{d=2}^{p-2} d^{-1}(x_0+1)(x_0+2) \cdots (x_0+d)x_1(x_1-1) \cdots (x_1-(p-d)+1) \\ &\quad + (x_0+1)x_1^2(x_1-1) \cdots (x_1-(p-2)) - (x_0+1)(x_0+2) \cdots (x_0+p-1)x_1^2 \end{aligned}$$

and, for the last two terms above, we have

$$\begin{aligned} (x_0+1)x_1^2(x_1-1) \cdots (x_1-(p-2)) &\equiv -(x_0+1)x_1(x_1-1) \cdots (x_1-(p-2)) , \\ (x_0+1)(x_0+2) \cdots (x_0+p-1)x_1^2 &\equiv (p-1)! \cdot \delta_0(x_0)x_1^2 = -(1-x_0^{p-1})x_1^2 \end{aligned}$$

where we used $x_1^2 \equiv x_1((x_1-(p-1))-1)$ and Wilson's Theorem $(p-1)! \equiv -1 \pmod{p}$.

By combining these results to Lemma 5.4, we have

$$\begin{aligned} \max(x_0, x_1) &= x_0 - x_0 \operatorname{argmax}(x_0, x_1) + x_1 \operatorname{argmax}(x_0, x_1) \\ &\equiv (x_1 - x_0) \sum_{d=2}^{p-2} d^{-1}(x_0+1)(x_0+2) \cdots (x_0+d)x_1(x_1-1) \cdots (x_1-(p-d)+1) \\ &\quad + x_0 - x_0(x_0+1)x_1(x_1-1) \cdots (x_1-(p-2)) \\ &\quad - (x_0+1)x_1(x_1-1) \cdots (x_1-(p-2)) + (1-x_0^{p-1})x_1^2 \\ &= (x_1 - x_0) \sum_{d=2}^{p-2} d^{-1}(x_0+1)(x_0+2) \cdots (x_0+d)x_1(x_1-1) \cdots (x_1-(p-d)+1) \\ &\quad + x_0 - (x_0+1)^2 x_1(x_1-1) \cdots (x_1-(p-2)) + (1-x_0^{p-1})x_1^2 \end{aligned}$$

and, for the second last term above, we have

$$(x_0+1)^2 x_1(x_1-1) \cdots (x_1-(p-2)) \equiv (x_0+1)^2 \cdot (p-1)! \cdot \delta_{p-1}(x_1) = -(x_0+1)^2(1-(x_1+1)^{p-1})$$

where we used Wilson's Theorem again. Hence, we have

$$\begin{aligned} \max(x_0, x_1) &\equiv (x_1 - x_0) \sum_{d=2}^{p-2} d^{-1}(x_0+1)(x_0+2) \cdots (x_0+d)x_1(x_1-1) \cdots (x_1-(p-d)+1) \\ &\quad + x_0 + (x_0+1)^2(1-(x_1+1)^{p-1}) + (1-x_0^{p-1})x_1^2 \end{aligned}$$

which is our claim in the statement. \square

6. POLYNOMIAL EXPRESSIONS OF SOME OTHER FUNCTIONS

In this section, we study the following two \mathbb{F}_p -valued functions that are related to max and argmax:

$$\begin{aligned} \text{ismax}(y; x) &= \chi(\max(x) = y) , \\ \text{nummax}^{(r)}(x) &= \#\{x_i \mid \max(x) = x_i\}^{(r)} , \end{aligned}$$

where $x \in \mathbb{F}_p^n$ and $y \in \mathbb{F}_p$. In practical applications, these functions are useful if there are “ties” in the vote.

By a careful interpretation of the definitions, we obtain minimal polynomials of these functions (which, however, consist of a lot of terms):

Proposition 6.1. Using the notation from §2, the following are minimal polynomial expressions:

$$\begin{aligned} \text{ismax}(y; x) &= \sum_{t=0}^{p-1} \delta_t(y) \sum_{i=0}^{n-1} \left(\prod_{j<i} L_t(x_j) \cdot \delta_t(x_i) \cdot \prod_{k>i} L_{t+1}(x_k) \right) , \\ \text{nummax}^{(0)}(x) &= \sum_{i=0}^{n-1} \chi(\max(x) = x_i) \\ &= \sum_{i=0}^{n-1} \sum_{0 \leq t \leq p-1} \left(\delta_t(x_i) \prod_{j \neq i} L_{t+1}(x_j) \right) , \\ \text{nummax}^{(r)}(x) &= \sum_{k=1}^n k^{(r)} \cdot \chi(\#\{i \mid \max(x) = x_i\} = k) \\ &= \sum_{k=1}^n k^{(r)} \left(\sum_{I \in \binom{[n]}{k}} \sum_{0 \leq t \leq p-1} \left(\prod_{i \in I} \delta_t(x_i) \prod_{j \notin I} L_{t+1}(x_j) \right) \right) . \end{aligned}$$

Proof. For the function ismax , given a constant $t \in \mathbb{F}_p$, we have $\max(x) = t$ if and only if there is an index i satisfying that $x_j < t$ for every $j < i$, $x_i = t$, and $x_k \leq t$ for every $k > i$; such an index i is unique if exists. This observation (in particular, the uniqueness of i) implies our claim.

For the function $\text{nummax}^{(0)}$, the function value is obtained by first counting the number of indices i with $\max(x) = x_i$ (or equivalently, $\chi(\max(x) = x_i) = 1$) and then taking the remainder of the number modulo p (i.e., just considering the number in \mathbb{F}_p). Moreover, given a constant $t \in \mathbb{F}_p$, we have $\max(x) = x_i = t$ if and only if $x_i = t$ and $x_j \leq t$ for every $j \neq i$. This observation implies our claim.

For the function $\text{nummax}^{(r)}$, given an integer $k \geq 1$ and a constant $t \in \mathbb{F}_p$, we have $\max(x) = t$ and $\#\{i \mid \max(x) = x_i\} = k$ if and only if there is a k -element set I of indices satisfying that $x_i = t$ for every $i \in I$ and $x_j < t$ for every $j \notin I$; such a set I is unique if exists. This observation (in particular, the uniqueness of I) implies our claim (note that $0^{(r)} = 0$ for any r). \square

When $p = 2$ and 3 , we give the following explicit minimal polynomial expressions of $\text{ismax}(y; x)$:

Proposition 6.2. When $p = 2$, a minimal polynomial expression of $\text{ismax}(y; x)$ is given by

$$\text{ismax}(y; x) = y + \prod_{i=0}^{n-1} (1 + x_i) .$$

When $p = 3$, a minimal polynomial expression of $\text{ismax}(y; x)$ is given by

$$\text{ismax}(y; x) = -y^2 + y \left(\prod_{i=0}^{n-1} (1 + x_i)^2 + \prod_{i=0}^{n-1} (1 - x_i^2) + 1 \right) + \prod_{i=0}^{n-1} (1 - x_i^2) .$$

Proof. First, we note that $\text{ismax}(y; x) = 1 - (y - \max(x))^{p-1}$ by the definition of the function. When $p = 2$, the right-hand side becomes $y + \max(x) + 1$ and now the claim follows from Proposition 3.2.

On the other hand, when $p = 3$, we have

$$\text{ismax}(y; x) = 1 - (y - \max(x))^2 = -y^2 - y \max(x) + 1 - \max(x)^2 .$$

Now we have $1 - \max(x)^2 = 1$ if $x_i = 0$ for all i , and $= 0$ otherwise. This implies that

$$1 - \max(x)^2 = \prod_{i=0}^{n-1} \delta_0(x_i) = \prod_{i=0}^{n-1} (1 - x_i^2)$$

and now the claim follows from Proposition 3.3. \square

Example 6.3. When $p = 2$, a minimal polynomial expression of $\text{nummax}^{(r)}(x)$ is given by

$$\text{nummax}^{(r)}(x) = e_{2^r} + n^{(r)} \prod_{i=0}^{n-1} (1 - x_i) .$$

This can be seen by the following argument. When $\max(x) = 0$, i.e., $x_i = 0$ for all i , we have $\text{nummax}^{(r)} = n^{(r)}$ for any r , which accounts for the second term. As $(\sum_{i=0}^{n-1} x_i)^{(r)} \equiv e_{2^r}(x) \pmod{2}$ by the result of [1] (see also [5, Example 1]), we obtain the equation.

7. FUTURE SUBJECT: MULTI-DIGIT CASE

We note that the previous sections studied functions with single-digit input values taken from \mathbb{F}_p ; in such a formulation, to handle larger input values we have to choose a larger prime p as well, which will result in polynomial expressions of the functions with higher degrees and much more involved structures. Another option to handle larger values is to express the input values in *multi-digit* forms; now each component of the input is identified with its p -ary expansion, therefore the entire input is regarded as a two-dimensional matrix over \mathbb{F}_p rather than a one-dimensional vector (over a larger field). In the latter model, the base field \mathbb{F}_p can be kept small even if the input values become larger. On the other hand, a large input value will then increase the total number of components of the input matrix, but this shortcoming might sometimes be avoidable in practice by implementation techniques such as parallel computation. This suggests that polynomial expressions of functions with multi-digit inputs are important as well.

However, even if the polynomial expression of a given function is understood well for single-digit input cases, it is in general a non-trivial task to deduce a polynomial expression of the function for multi-digit input cases. We leave such multi-digit extensions of the results in this paper as a future research topic, and we just conclude this paper with an example:

Proposition 7.1. Let $p = 2$, and consider two-bit inputs $y = 2y_1 + y_0 \in \{0, 1, 2, 3\}$ and $x_i = 2x_{i,1} + x_{i,0} \in \{0, 1, 2, 3\}$ for $0 \leq i \leq n-1$, where $y_j, x_{i,j} \in \mathbb{F}_2$. Then the following is a minimal polynomial expression:

$$\begin{aligned} \text{ismax}(y; x) &= \text{ismax}(y; x_0, x_1, \dots, x_{n-1}) \\ &= y_1 y_0 + y_1 \prod_{i=0}^{n-1} (1 + x_{i,1} x_{i,0}) + (y_1 + y_0) \prod_{i=0}^{n-1} (1 + x_{i,1}) + (y_1 + 1) \prod_{i=0}^{n-1} (1 + x_{i,1})(1 + x_{i,0}) . \end{aligned}$$

Proof. As the right-hand side of the statement satisfies the minimality conditions for the degrees, it suffices to verify that the values of both terms are equal for any input values.

First we note that, for any set I of index pairs (i, j) , we have

$$\prod_{(i,j) \in I} (1 + x_{i,j}) = \chi(x_{i,j} = 0 \text{ for all } (i, j) \in I) .$$

Similarly, we have

$$\prod_i (1 + x_{i,1} x_{i,0}) = \chi(\text{for any } i, \text{ either } x_{i,1} = 0 \text{ or } x_{i,0} = 0 \text{ holds}) .$$

We divide the argument according to the values of y_1 and y_0 . When $y_1 = y_0 = 0$, we have $\text{ismax}(y; x) = 1$ if and only if $x_{i,1} = x_{i,0} = 0$ for every index i . Now the right-hand side of the statement becomes $\prod_{i=0}^{n-1} (1 + x_{i,1})(1 + x_{i,0})$, which coincides with $\text{ismax}(y; x)$ by the remark above.

When $y_1 = 0$ and $y_0 = 1$, the right-hand side of the statement becomes $\prod_{i=0}^{n-1} (1 + x_{i,1}) + \prod_{i=0}^{n-1} (1 + x_{i,1})(1 + x_{i,0})$. Now if at least one of $x_{i,1}$ is 1, then we have $\text{ismax}(y; x) = 0$ by definition, while the value

of the polynomial becomes 0 as well by the remark above, as desired. In the remaining case where $x_{i,1} = 0$ for every i , we have $\text{ismax}(y; x) = 1$ if and only if $x_{i,0} = 1$ for some i ; while the polynomial now becomes $1 + \prod_{i=0}^{n-1} (1 + x_{i,0})$. By the remark above, the value of the polynomial coincides with $\text{ismax}(y; x)$, as desired.

When $y_1 = 1$ and $y_0 = 0$, the right-hand side of the statement becomes $\prod_{i=0}^{n-1} (1 + x_{i,1} x_{i,0}) + \prod_{i=0}^{n-1} (1 + x_{i,1})$. Now if $x_{i,1} = 0$ for every i , then we have $\text{ismax}(y; x) = 0$ by definition, while the value of the polynomial becomes $1 + 1 = 0$ as well by the remark above, as desired. In the remaining case where $x_{i,1} = 1$ for some i , let I denote the set of indices i with $x_{i,1} = 1$ (hence now $I \neq \emptyset$). In this case, we have $\text{ismax}(y; x) = 1$ if and only if $x_{i,0} = 0$ for every $i \in I$; while the polynomial now becomes $\prod_{i \in I} (1 + x_{i,0})$. By the remark above, the value of the polynomial coincides with $\text{ismax}(y; x)$, as desired.

Finally, when $y_1 = y_0 = 1$, the right-hand side of the statement becomes $1 + \prod_{i=0}^{n-1} (1 + x_{i,1} x_{i,0})$. By the remark above, this polynomial takes the value 1 if and only if $x_{i,1} = x_{i,0} = 1$ for some index i ; this condition is precisely the same as the condition for $\text{ismax}(y; x)$ in the present case to take the value 1, by definition. This completes the proof. \square

REFERENCES

- [1] J. Boyar, R. Peralta, D. Pochuev: On the Multiplicative Complexity of Boolean Functions over the Basis $(\text{cap}, +, 1)$. Theoretical Computer Science, vol.235(1) (2000) pp.43–57.
- [2] J. H. Cheon, M. Kim, M. Kim: Search-and-Compute on Encrypted Data. In: Proceedings of Financial Cryptography and Data Security 2015 (FC 2015), Springer Lecture Notes in Computer Science vol.8976, 2015, pp.142–159.
- [3] C. Gentry: Computing on the Edge of Chaos: Structure and Randomness in Encrypted Computation. The 2014 International Congress of Mathematicians (ICM 2014), Seoul, Korea, August 16, 2014. Proceedings available at: IACR Cryptology ePrint Archive, report 2014/610, <http://eprint.iacr.org/2014/610> (2014).
- [4] C. Gentry: Fully Homomorphic Encryption Using Ideal Lattices. In: Proceedings of STOC 2009, ACM, 2009, pp.169–178.
- [5] S. Kaji, T. Maeno, K. Nuida, Y. Numata: Polynomial Expressions of Carries in p -ary Arithmetics. Preprint, arXiv:1506.02742 (2015).
- [6] K. Nuida, K. Kurosawa: (Batch) Fully Homomorphic Encryption over Integers for Non-Binary Message Spaces. In: Proceedings of EUROCRYPT 2015 (Part I), Springer Lecture Notes in Computer Science vol.9056, 2015, pp.537–555.
- [7] A. Silverberg: Fully Homomorphic Encryption for Mathematicians. IACR Cryptology ePrint Archive, report 2013/250, <http://eprint.iacr.org/2013/250> (2013).
- [8] C. Sturttivant, G. S. Frandsen: The Computational Efficacy of Finite-Field Arithmetic. Theoretical Computer Science, vol.112 (1993) pp.291–309.

(Shizuo Kaji) YAMAGUCHI UNIVERSITY, JAPAN
JAPAN SCIENCE AND TECHNOLOGY AGENCY (JST) PRESTO RESEARCHER
E-mail address: skaji@yamaguchi-u.ac.jp

(Toshiaki Maeno) MEIJO UNIVERSITY, JAPAN
E-mail address: tmaeno@meijo-u.ac.jp

(Koji Nuida) NATIONAL INSTITUTE OF ADVANCED INDUSTRIAL SCIENCE AND TECHNOLOGY (AIST), JAPAN
JAPAN SCIENCE AND TECHNOLOGY AGENCY (JST) PRESTO RESEARCHER
E-mail address: k.nuida@aist.go.jp

(Yasuhide Numata) SHINSHU UNIVERSITY, JAPAN
E-mail address: nu@math.shinshu-u.ac.jp